Lecture Notes to Accompany

Scientific Computing

An Introductory Survey

Second Edition

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Chapter 7

Interpolation

Interpolation

Basic interpolation problem: for given data

$$(t_i, y_i), \quad i = 1, \ldots, m,$$

with $t_1 < t_2 < \ldots < t_m$, determine function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(t_i) = y_i, \quad i = 1, \ldots, m$$

f is *interpolating function*, or *interpolant*, for given data

Additional data might be prescribed, such as slope of interpolant at given points

Additional constraints might be imposed, such as smoothness, monotonicity, or convexity of interpolant

f could be function of more than one variable, but we will consider only one-dimensional case

Purposes for Interpolation

- Plotting smooth curve through discrete data points
- Reading between lines of table
- Differentiating or integrating tabular data
- Quick and easy evaluation of mathematical function
- Replacing complicated function by simple one

Interpolation vs Approximation

By definition, interpolating function fits given data points exactly

Interpolation inappropriate if data points subject to significant errors

Usually preferable to smooth noisy data, for example by least squares approximation

Approximation also appropriate for special function libraries

Issues in Interpolation

Arbitrarily many functions interpolate given data points

- What form should function have?
- How should function behave between data points?
- Should function inherit properties of data, such as monotonicity, convexity, or periodicity?
- Are parameters that define interpolating function meaningful?
- If function and data are plotted, should results be visually pleasing?

Choosing Interpolant

Choice of function for interpolation based on

- How easy function is to work with
  - determining its parameters
  - evaluating function
  - differentiating or integrating function
- How well properties of function match properties of data to be fit (smoothness, monotonicity, convexity, periodicity, etc.)

Functions for Interpolation

Families of functions commonly used for interpolation include

- Polynomials
- Piecewise polynomials
- Trigonometric functions
- Exponential functions
- Rational functions

We will focus on interpolation by polynomials and piecewise polynomials for now

Will consider trigonometric interpolation (DFT) later

Basis Functions

Family of functions for interpolating given data points is spanned by set of *basis functions* $\phi_1(t)$, $\ldots, \phi_n(t)$

Interpolating function f chosen as linear combination of basis functions,

$$f(t) = \sum_{j=1}^{n} x_j \phi_j(t)$$

Requiring f to interpolate data (t_i, y_i) means

$$f(t_i) = \sum_{j=1}^{n} x_j \phi_j(t_i) = y_i, \quad i = 1, \ldots, m,$$

which is system of linear equations

$$\boldsymbol{Ax} = \boldsymbol{y}$$

for n-vector $\boldsymbol{x}$ of parameters x_j, where entries of $m \times n$ matrix $\boldsymbol{A}$ are given by $a_{ij} = \phi_j(t_i)$

Existence, Uniqueness, and Conditioning

Existence and uniqueness of interpolant depend on number of data points m and number of basis functions n

If $m > n$, interpolant usually doesn't exist

If $m < n$, interpolant not unique

If $m = n$, then basis matrix $\boldsymbol{A}$ nonsingular, provided data points t_i distinct, so data can be fit exactly

Sensitivity of parameters $\boldsymbol{x}$ to perturbations in data depends on cond($\boldsymbol{A}$), which depends in turn on choice of basis functions

Polynomial Interpolation

Simplest and most common type of interpolation uses polynomials

Unique polynomial of degree at most $n - 1$ passes through n data points (t_i, y_i), $i = 1, \ldots, n$, where t_i are distinct

There are many ways to represent or compute polynomial, but in theory all must give same result

Monomial Basis

Monomial basis functions,

$$\phi_j(t) = t^{j-1}, \quad j = 1, \dots, n,$$

give interpolating polynomial of form

$$p_{n-1}(t) = x_1 + x_2 t + \cdots + x_n t^{n-1},$$

with coefficients $\boldsymbol{x}$ given by $n \times n$ linear system

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ 1 & t_2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{y}$$

Matrix of this form called *Vandermonde* matrix

Example: Monomial Basis

Find polynomial of degree two interpolating three data points $(-2,-27)$, $(0,-1)$, $(1,0)$

Using monomial basis, linear system is

$$\boldsymbol{Ax} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boldsymbol{y}$$

For these particular data, system is

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -27 \\ -1 \\ 0 \end{bmatrix},$$

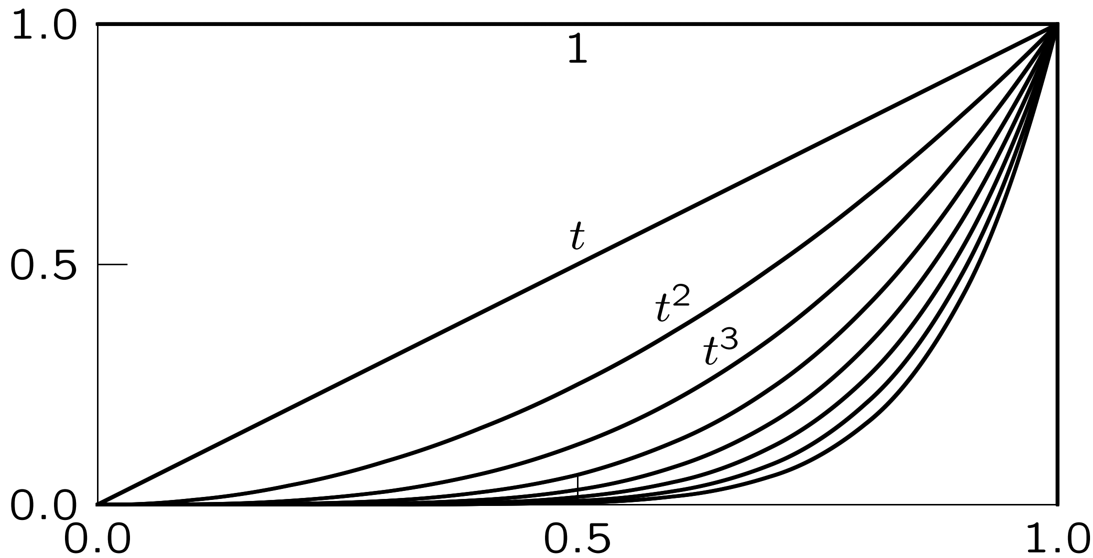
whose solution is $x = [\,-1 \quad 5 \quad -4\,]^T$, so interpolating polynomial is

$$p_2(t) = -1 + 5t - 4t^2$$

Monomial Basis, continued

Solving system $\boldsymbol{Ax} = \boldsymbol{y}$ using standard linear equation solver to determine coefficients $\boldsymbol{x}$ of interpolating polynomial requires $\mathcal{O}(n^3)$ work

For monomial basis, matrix $\boldsymbol{A}$ often ill-conditioned, especially for high-degree polynomials



Monomial Basis, continued

Ill-conditioning does not prevent fitting data points well, since residual for linear system solution will be small

But it does mean that values of coefficients may be poorly determined

Both conditioning of linear system and amount of computational work required to solve it can be improved by using different basis

Change of basis still gives same interpolating polynomial for given data, but *representation* of polynomial will be different

Monomial Basis, continued

Conditioning with monomial basis can be improved by shifting and scaling independent variable t:

$$\phi_j(t) = \left(\frac{t-c}{d}\right)^{j-1},$$

where, $c = (t_1 + t_n)/2$ is midpoint and $d = (t_n - t_1)/2$ is half of range of data

New independent variable lies in interval $[-1, 1]$, which also helps avoid overflow or harmful underflow

Even with optimal shifting and scaling, monomial basis usually still poorly conditioned, and we seek better alternatives

Evaluating Polynomials

When represented in monomial basis, polynomial

$$p_{n-1}(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$$

can be evaluated efficiently using *Horner's* nested evaluation scheme:

$$p_{n-1}(t) = x_1 + t(x_2 + t(x_3 + t(\cdots(x_{n-1} + t x_n)\cdots))),$$

which requires only n additions and n multiplications

For example,

$$1 - 4t + 5t^2 - 2t^3 + 3t^4 = 1 + t(-4 + t(5 + t(-2 + 3t)))$$

Other manipulations of interpolating polynomial, such as differentiation or integration, are also relatively easy with monomial basis representation

Lagrange Interpolation

For given set of data points (t_i, y_i), $i = 1, \ldots, n$, *Lagrange* basis functions given by

$$\ell_j(t) = \prod_{k=1,k\neq j}^{n} (t - t_k) \;/\; \prod_{k=1,k\neq j}^{n} (t_j - t_k),$$

$j = 1, \ldots, n$

For Lagrange basis,

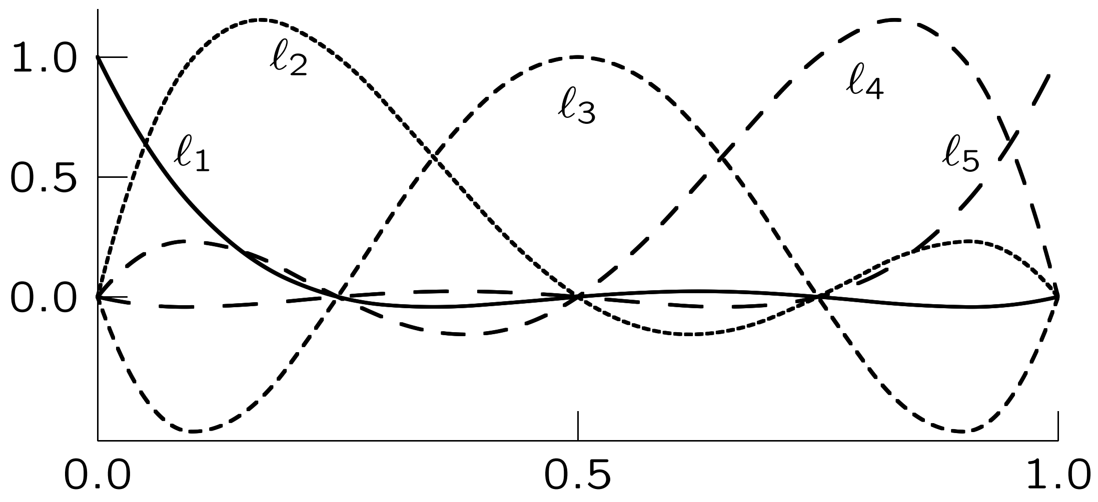
$$\ell_j(t_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad i, j = 1, \ldots, n$$

so matrix of linear system $\boldsymbol{Ax} = \boldsymbol{y}$ is identity

Thus, Lagrange polynomial interpolating data points (t_i, y_i) given by

$$p_{n-1}(t) = y_1\ell_1(t) + y_2\ell_2(t) + \cdots + y_n\ell_n(t)$$

Lagrange Basis Functions



Lagrange interpolant is easy to determine but more expensive to evaluate for given argument, compared with monomial basis representation

Lagrangian form is also more difficult to differentiate, integrate, etc.

Example: Lagrange Interpolation

Use Lagrange interpolation to find interpolating polynomial for three data points $(-2, -27)$, $(0, -1)$, $(1, 0)$

Lagrange polynomial of degree two interpolating three points (t_1, y_1), (t_2, y_2), (t_3, y_3) is

$$p_2(t) = y_1 \frac{(t - t_2)(t - t_3)}{(t_1 - t_2)(t_1 - t_3)} + y_2 \frac{(t - t_1)(t - t_3)}{(t_2 - t_1)(t_2 - t_3)} + y_3 \frac{(t - t_1)(t - t_2)}{(t_3 - t_1)(t_3 - t_2)}$$

For these particular data, this becomes

$$p_2(t) = -27 \frac{t(t - 1)}{(-2)(-2 - 1)} + (-1) \frac{(t + 2)(t - 1)}{(2)(-1)}$$

Newton Interpolation

For given set of data points (t_i, y_i), $i = 1, \ldots, n$, *Newton* basis functions given by

$$\pi_j(t) = \prod_{k=1}^{j-1} (t - t_k), \quad j = 1, \ldots, n,$$

where value of product taken to be 1 when limits make it vacuous

Newton interpolating polynomial has form

$$p_{n-1}(t) = x_1 + x_2(t - t_1) + x_3(t - t_1)(t - t_2) + \cdots$$
$$+ x_n(t - t_1)(t - t_2) \cdots (t - t_{n-1})$$

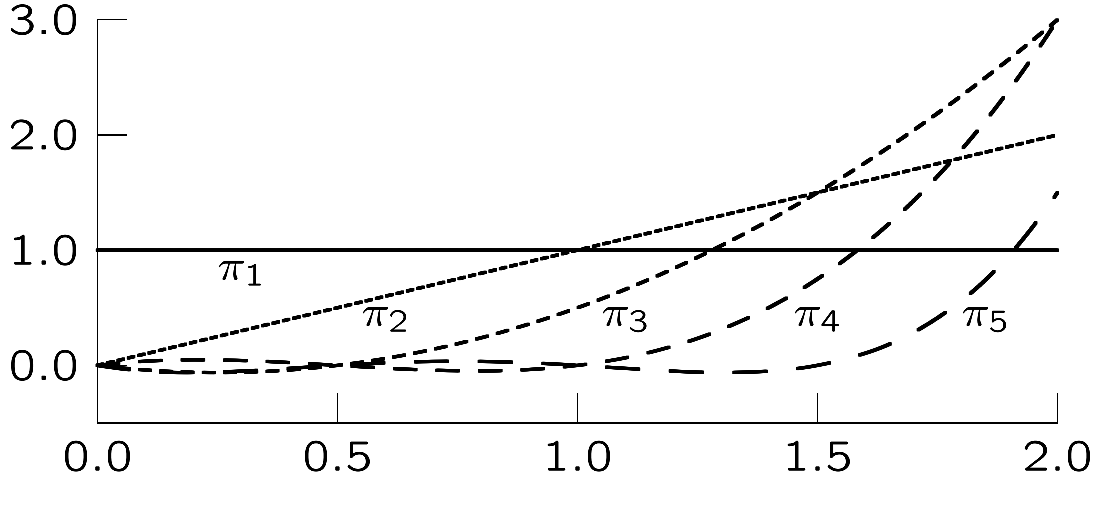
For $i < j$, $\pi_j(t_i) = 0$, so basis matrix $\boldsymbol{A}$ is lower triangular, where $a_{ij} = \pi_j(t_i)$

Newton Interpolation, continued

Hence, solution $\boldsymbol{x}$ to system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ can be computed by forward-substitution in $\mathcal{O}(n^2)$ arithmetic operations

Moreover, resulting interpolant can be evaluated efficiently for any argument by nested evaluation scheme similar to Horner's method

Newton interpolation has better balance between cost of computing interpolant and cost of evaluating it



Example: Newton Interpolation

Use Newton interpolation to find interpolating polynomial for three data points $(-2,-27)$, $(0,-1)$, $(1,0)$

Using Newton basis, linear system is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & t_2 - t_1 & 0 \\ 1 & t_3 - t_1 & (t_3 - t_1)(t_3 - t_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

For these particular data, system is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -27 \\ -1 \\ 0 \end{bmatrix},$$

whose solution by forward substitution is $x = [\,-27 \quad 13 \quad -4\,]^T$, so interpolating polynomial is

$$p(t) = -27 + 13(t+2) - 4(t+2)t$$

Newton Interpolation, continued

If $p_j(t)$ is polynomial of degree $j-1$ interpolating j given points, then for any constant x_{j+1},

$$p_{j+1}(t) = p_j(t) + x_{j+1}\pi_{j+1}(t)$$

is polynomial of degree j that also interpolates same j points

Free parameter x_{j+1} can then be chosen so that $p_{j+1}(t)$ interpolates y_{j+1}. Specifically,

$$x_{j+1} = \frac{y_{j+1} - p_j(t_{j+1})}{\pi_{j+1}(t_{j+1})}$$

Newton interpolation begins with constant polynomial $p_1(t) = y_1$ interpolating first data point and then successively incorporates remaining data points into interpolant

Divided Differences

Given data points (t_i, y_i), $i = 1, \ldots, n$, *divided differences*, denoted by $f[\,]$, defined recursively by

$$f[t_1, t_2, \ldots, t_k] = \frac{f[t_2, t_3, \ldots, t_k] - f[t_1, t_2, \ldots, t_{k-1}]}{t_k - t_1},$$

where recursion begins with $f[t_k] = y_k$, $k = 1, \ldots, n$

Coefficient of jth basis function in Newton interpolant given by

$$x_j = f[t_1, t_2, \ldots, t_j]$$

Recursion requires $\mathcal{O}(n^2)$ arithmetic operations to compute coefficients of Newton interpolant, but is less prone to overflow or underflow than direct formation of triangular Newton basis matrix

Orthogonal Polynomials

Inner product can be defined on space of polynomials on interval $[a, b]$ by taking

$$\langle p, q \rangle = \int_a^b p(t)q(t)w(t)dt,$$

where $w(t)$ is nonnegative *weight function*

Two polynomials p and q are *orthogonal* if $\langle p, q \rangle = 0$

Set of polynomials $\{p_i\}$ is *orthonormal* if

$$\langle p_i, p_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Given set of polynomials, Gram-Schmidt orthogonalization can be used to generate orthonormal set spanning same space

Orthogonal Polynomials, continued

For example, with inner product given by weight function $w(t) \equiv 1$ on interval $[-1, 1]$, applying Gram-Schmidt process to set of monomials $1, t, t^2, t^3, \ldots$ yields *Legendre* polynomials

$$1, \quad t, \quad (3t^2 - 1)/2, \quad (5t^3 - 3t)/2,$$

$$(35t^4 - 30t^2 + 3)/8, \quad (63t^5 - 70t^3 + 15t)/8, \quad \ldots,$$

first n of which form an orthogonal basis for space of polynomials of degree at most $n - 1$

Other choices of weight functions and intervals yield other orthogonal polynomials, such as Chebyshev, Jacobi, Laguerre, and Hermite

Orthogonal Polynomials, continued

Orthogonal polynomials have many useful properties

They satisfy three-term recurrence relation of form

$$p_{k+1}(t) = (\alpha_k t + \beta_k) p_k(t) - \gamma_k p_{k-1}(t),$$

which makes them very efficient to generate and evaluate

Orthogonality makes them very natural for least squares approximation, and they are also useful for generating Gaussian quadrature rules

Chebyshev Polynomials

kth *Chebyshev* polynomial of first kind defined on interval $[-1, 1]$ by

$$T_k(t) = \cos(k \arccos(t))$$

are orthogonal with respect to weight function $(1 - t^2)^{-1/2}$

First few Chebyshev polynomials given by

$$1, \quad t, \quad 2t^2 - 1, \quad 4t^3 - 3t, \quad 8t^4 - 8t^2 + 1,$$

$$16t^5 - 20t^3 + 5t, \quad \ldots.$$

Equi-oscillation property: successive extrema of T_k are equal in magnitude and alternate in sign, which distributes error uniformly when approximating arbitrary continuous function

Chebyshev Points

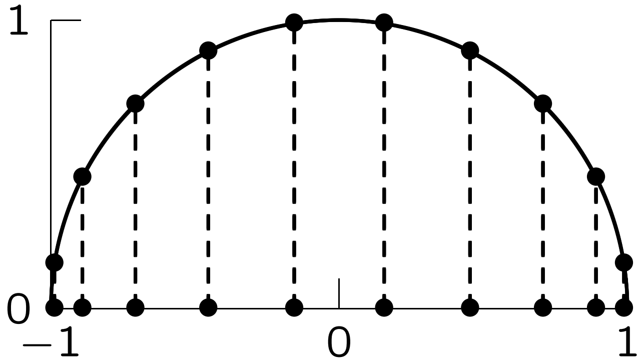
Chebyshev points are zeros of T_k, given by

$$t_i = \cos\left(\frac{(2i-1)\pi}{2k}\right), \quad i = 1, \ldots, k,$$

or extrema of T_k, given by

$$t_i = \cos\left(\frac{i\pi}{k}\right), \quad i = 0, 1, \ldots, k.$$

Chebyshev points are abscissas of points equally spaced around unit circle in plane $\mathbb{R}^2$



Chebyshev points have attractive properties for interpolation and other problems

Interpolating Continuous Functions

If data points are discrete sample of continuous function, how well does interpolant approximate that function between sample points?

If f is smooth function, and p_{n-1} is polynomial of degree at most $n-1$ interpolating f at n points $t_1, \ldots, t_n$, then

$$f(t) - p_{n-1}(t) = \frac{f^{(n)}(\theta)}{n!}(t-t_1)(t-t_2)\cdots(t-t_n),$$

where θ is some (unknown) point in interval $[t_1, t_n]$

Since point θ unknown, result not particularly useful unless bound on appropriate derivative of f is known

Interpolating Continuous Functions, cont.

If $|f^{(n)}(t)| \leq M$ for all $t \in [t_1, t_n]$, and $h = \max\{t_{i+1} - t_i : i = 1, \ldots, n-1\}$, then

$$\max_{t \in [t_1, t_n]} |f(t) - p_{n-1}(t)| \leq \frac{Mh^n}{4n}$$

Error diminishes with increasing n and decreasing h, but only if $|f^{(n)}(t)|$ does not grow too rapidly with n

High-Degree Polynomial Interpolation

Interpolating polynomials of high degree are expensive to determine and evaluate

In some bases, coefficients of polynomial may be poorly determined due to ill-conditioning of linear system to be solved

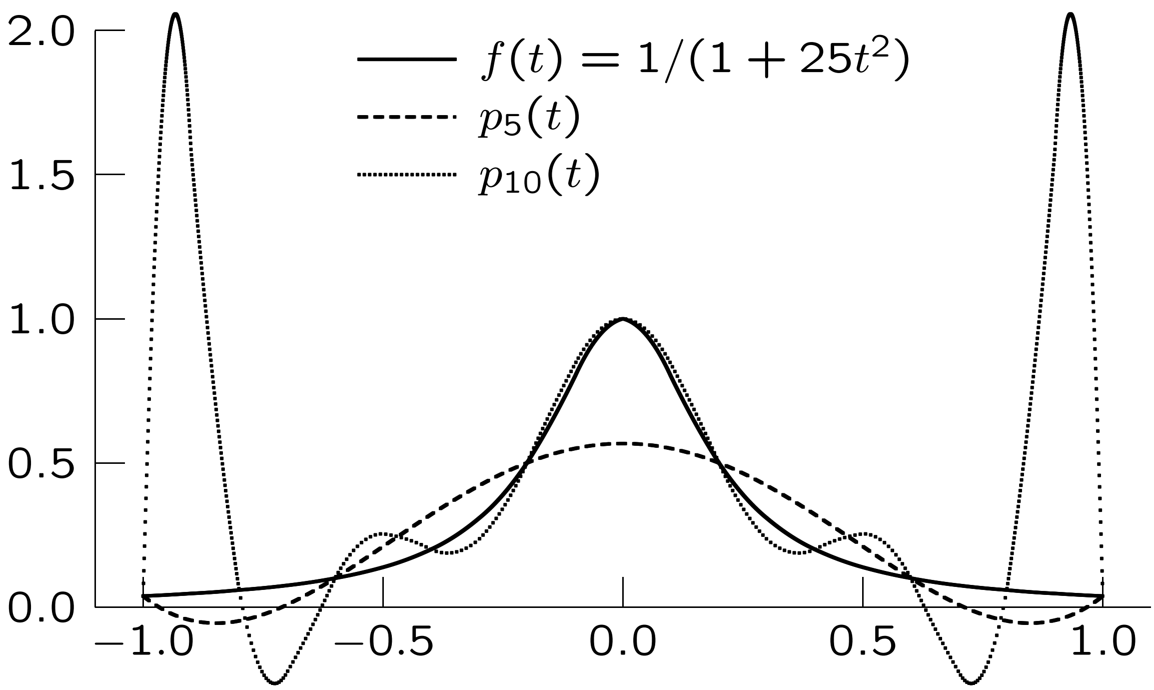
High-degree polynomial necessarily has lots of "wiggles," which may bear no relation to data to be fit

Polynomial goes through required data points, but it may oscillate wildly between data points

Nonconvergence

Polynomial interpolating continuous function at equally spaced points may not converge to function as number of data points and polynomial degree increases

Example: Polynomial interpolants of Runge's function at equally spaced points



Placement of Interpolation Points

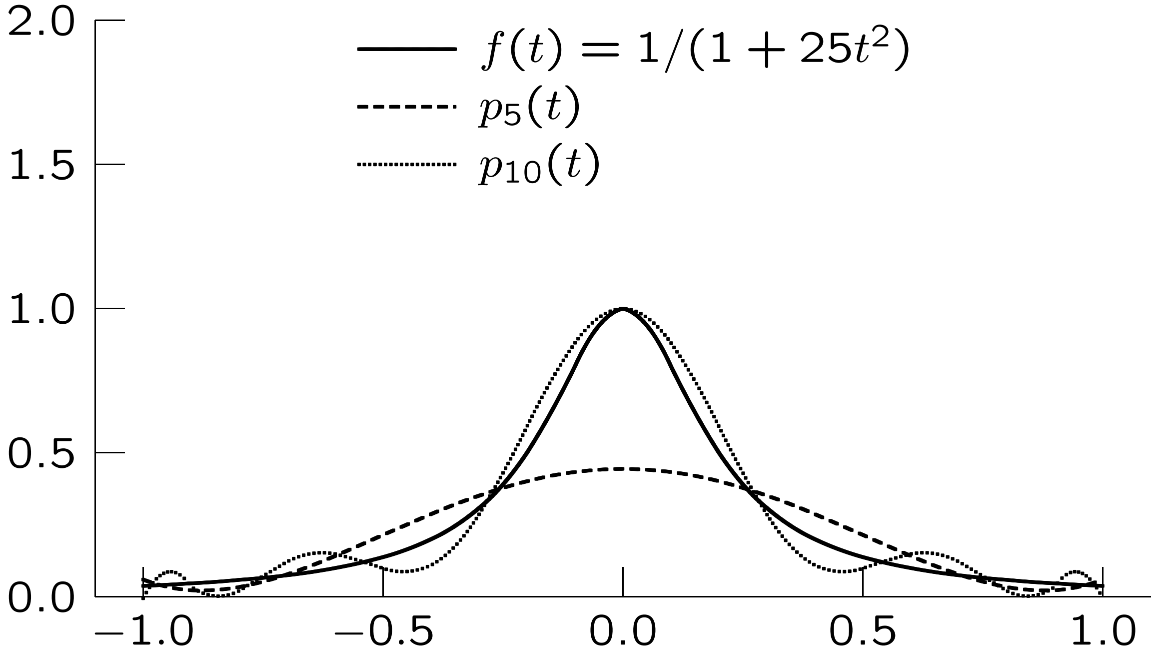
Equally spaced interpolation points often yield unsatisfactory results near ends of interval

If points are bunched near ends of interval, more satisfactory results likely to be obtained with polynomial interpolation

For example, use of Chebyshev points distributes error evenly and yields convergence throughout interval for any sufficiently smooth function

Placement of Points, continued

Example: Polynomial interpolants of Runge's function at Chebyshev points



Taylor Polynomial

Another useful form of polynomial interpolation for smooth function f is polynomial given by truncated Taylor series

$$p_n(t) = f(a) + f'(a)(t-a) + \frac{f''(a)}{2}(t-a)^2 + \cdots$$
$$+ \frac{f^{(n)}(a)}{n!}(t-a)^n$$

Polynomial interpolates f in that values of p_n and its first n derivatives match those of f and its first n derivatives evaluated at $t = a$, so $p_n(t)$ is good approximation to $f(t)$ for t near a

We have already seen examples in Newton's method for nonlinear equations and optimization

Piecewise Polynomial Interpolation

Fitting single polynomial to large number of data points is likely to yield unsatisfactory oscillating behavior in interpolant

Piecewise polynomials provide alternative to practical and theoretical difficulties with high-degree polynomial interpolation

Main advantage of piecewise polynomial interpolation is that large number of data points can be fit with low-degree polynomials

In piecewise interpolation of given data points (t_i, y_i), *different* function is used in each subinterval $[t_i, t_{i+1}]$

Abscissas t_i are called *knots* or *breakpoints*, at which interpolant changes from one function to another

Piecewise Interpolation, continued

Simplest example is piecewise linear interpolation, in which successive pairs of data points are connected by straight lines

Although piecewise interpolation eliminates excessive oscillation and nonconvergence, it appears to sacrifice smoothness of interpolating function

We have many degrees of freedom in choosing piecewise polynomial interpolant, however, which can be exploited to obtain smooth interpolating function despite its piecewise nature

Hermite Interpolation

In *Hermite* interpolation, derivatives as well as values of interpolating function are specified at data points

Specifying derivative values adds more equations to linear system that determines parameters of interpolating function

To have unique solution, number of equations must equal number of parameters to be determined

Piecewise cubic polynomials are typical choice Hermite interpolation, providing flexibility, simplicity and efficiency

Hermite Cubic Interpolation

Hermite cubic interpolant is piecewise cubic polynomial interpolant with continuous first derivative

Piecewise cubic polynomial with n knots has $4(n-1)$ parameters to be determined

Requiring that it interpolate given data gives $2(n-1)$ equations

Requiring that it have one continuous derivative gives $n-2$ additional equations, or total of $3n-4$, which still leaves n free parameters

Thus, Hermite cubic interpolant is not unique, and remaining free parameters can be chosen so that result satisfies additional constraints

Cubic Spline Interpolation

Spline is piecewise polynomial of degree k that is $k-1$ times continuously differentiable

For example, linear spline is of degree 1 and has 0 continuous derivatives, i.e., it is continuous, but not smooth, and could be described as "broken line"

Cubic spline is piecewise cubic polynomial that is twice continuously differentiable

As with Hermite cubic, interpolating given data and requiring one continuous derivative imposes $3n-4$ constraints on cubic spline

Requiring continuous second derivative imposes $n-2$ additional constraints, leaving 2 remaining free parameters

Cubic Splines, continued

Final two parameters can be fixed in various ways:

- Specifying first derivative at endpoints t_1 and t_n
- Forcing second derivative to be zero at endpoints, which gives *natural* spline
- Enforcing "not-a-knot" condition, forcing two consecutive cubic pieces to be same
- Forcing first derivatives, as well as second derivatives, to match at endpoints t_1 and t_n (if spline is to be periodic)

Example: Cubic Spline Interpolation

Determine natural cubic spline interpolating three data points (t_i, y_i), $i = 1, 2, 3$

Required interpolant is piecewise cubic function defined by separate cubic polynomials in each of two intervals $[t_1, t_2]$ and $[t_2, t_3]$

Denote these two polynomials by

$$p_1(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \alpha_4 t^3,$$

$$p_2(t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^3$$

Eight parameters are to be determined, so we need eight equations

Requiring first cubic to interpolate data at end points of first interval gives two equations

$$\alpha_1 + \alpha_2 t_1 + \alpha_3 t_1^2 + \alpha_4 t_1^3 = y_1,$$

Example Continued

$$\alpha_1 + \alpha_2 t_2 + \alpha_3 t_2^2 + \alpha_4 t_2^3 = y_2$$

Requiring second cubic to interpolate data at end points of second interval gives two equations

$$\beta_1 + \beta_2 t_2 + \beta_3 t_2^2 + \beta_4 t_2^3 = y_2,$$

$$\beta_1 + \beta_2 t_3 + \beta_3 t_3^2 + \beta_4 t_3^3 = y_3$$

Requiring first derivative of interpolant to be continuous at t_2 gives equation

$$\alpha_2 + 2\alpha_3 t_2 + 3\alpha_4 t_2^2 = \beta_2 + 2\beta_3 t_2 + 3\beta_4 t_2^2$$

Example Continued

Requiring second derivative of interpolant function to be continuous at t_2 gives equation

$$2\alpha_3 + 6\alpha_4 t_2 = 2\beta_3 + 6\beta_4 t_2$$

Finally, by definition natural spline has second derivative equal to zero at endpoints, which gives two equations

$$2\alpha_3 + 6\alpha_4 t_1 = 0,$$

$$2\beta_3 + 6\beta_4 t_3 = 0$$

When particular data values are substituted for t_i and y_i, system of eight linear equations can be solved for eight unknown parameters α_i and β_i

Hermite Cubic vs Spline Interpolation

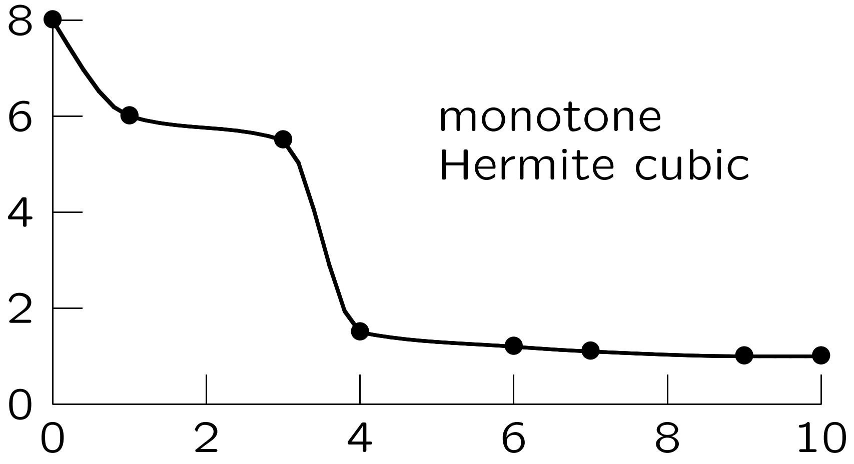
Choice between Hermite cubic and spline interpolation depends on data to be fit and on purpose for doing interpolation

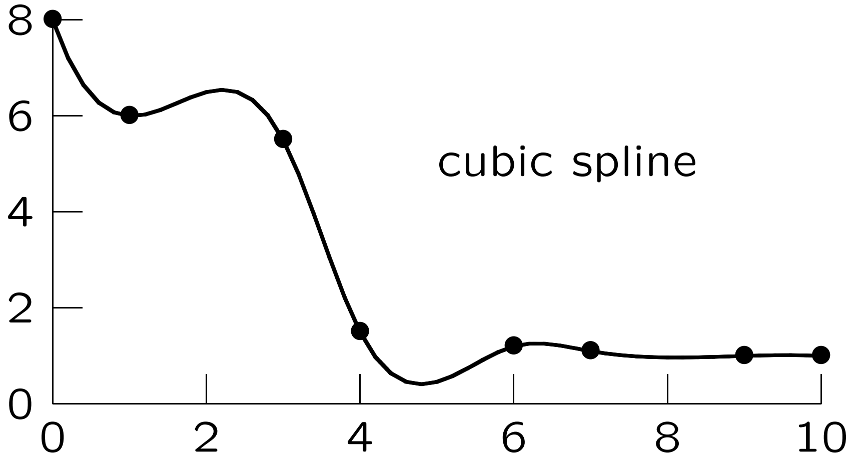
If smoothness is of paramount importance, then spline interpolation may be most appropriate

But Hermite cubic interpolant may have more pleasing visual appearance and allows flexibility to preserve monotonicity if original data are monotonic

In any case, it is advisable to plot interpolant and data to help assess how well interpolating function captures behavior of original data

Hermite Cubic vs Spline Interpolation





B-splines

B-splines form basis for family of spline functions of given degree

B-splines can be defined in various ways, including recursion, convolution, and divided differences. Here we will define them recursively

Although in practice we use only finite set of knots $t_1, \ldots, t_n$, for notational convenience we will assume infinite set of knots

$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots$$

Additional knots can be taken as arbitrarily defined points outside interval $[t_1, t_n]$

We will also use linear functions

$$v_i^k(t) = (t - t_i)/(t_{i+k} - t_i)$$

B-splines, continued

To start recursion, define B-splines of degree 0 by

$$B_i^0(t) = \begin{cases} 1 & \text{if } t_i \le t < t_{i+1} \\ 0 & \text{otherwise} \end{cases},$$

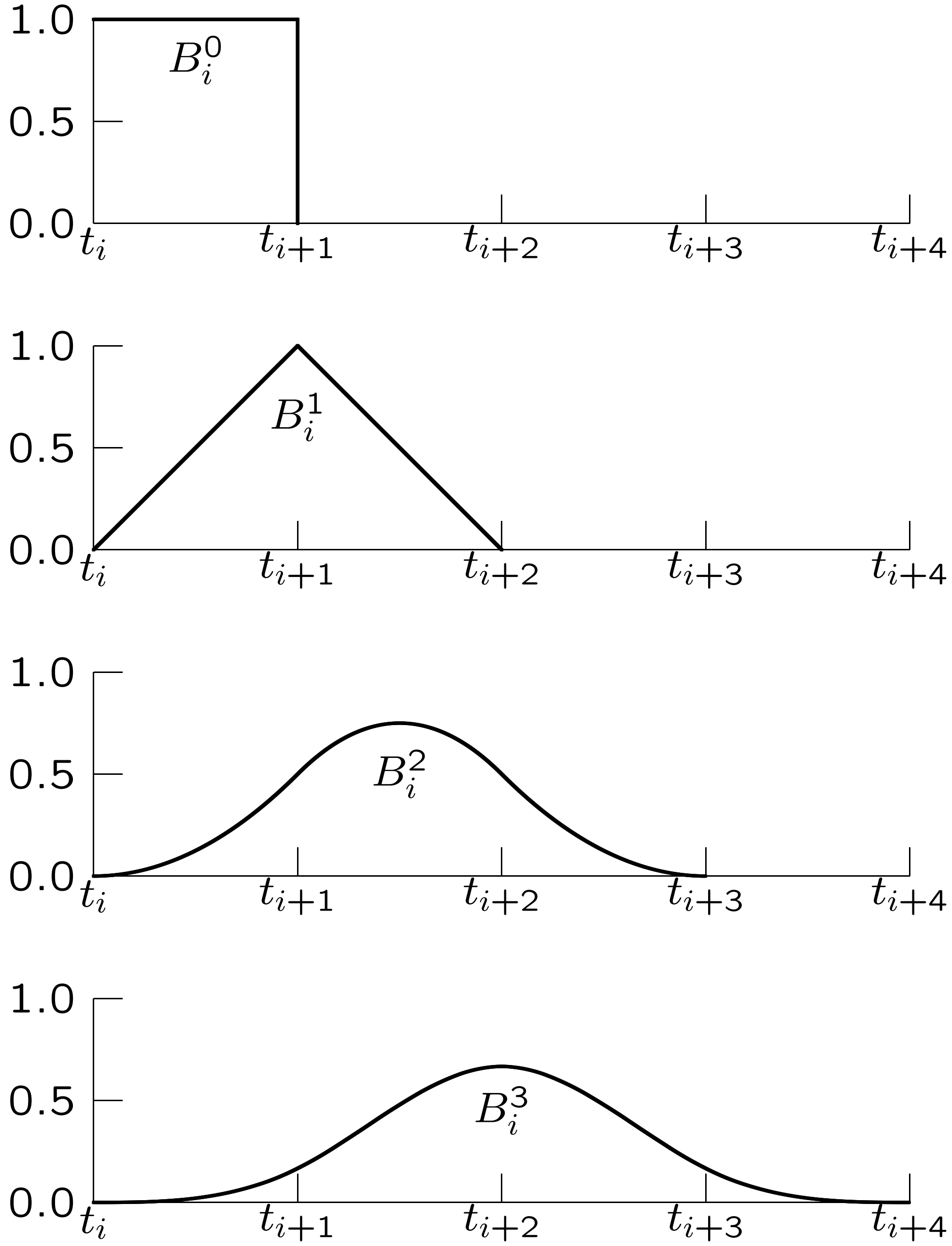
and then for $k > 0$ define B-splines of degree k by

$$B_i^k(t) = v_i^k(t) B_i^{k-1}(t) + (1 - v_{i+1}^k(t)) B_{i+1}^{k-1}(t)$$

Since B_i^0 is piecewise constant and v_i^k is linear, B_i^1 is piecewise linear

Similarly, B_i^2 is in turn piecewise quadratic, and in general, B_i^k is piecewise polynomial of degree k

B-splines


1.0
0.5
0.0
B_i^0
t_i
t_{i+1}
t_{i+2}
t_{i+3}
t_{i+4}
1.0
0.5
0.0
B_i^1
t_i
t_{i+1}
t_{i+2}
t_{i+3}
t_{i+4}
1.0
0.5
0.0
B_i^2
t_i
t_{i+1}
t_{i+2}
t_{i+3}
t_{i+4}
1.0
0.5
0.0
B_i^3
t_i
t_{i+1}
t_{i+2}
t_{i+3}
t_{i+4}

B-splines, continued

Important properties of B-spline functions B_i^k:

1. For $t < t_i$ or $t > t_{i+k+1}$, $B_i^k(t) = 0$

2. For $t_i < t < t_{i+k+1}$, $B_i^k(t) > 0$

3. For all t, $\sum_{i=-\infty}^{\infty} B_i^k(t) = 1$

4. For $k \geq 1$, B_i^k has $k-1$ continuous derivatives

5. Set of functions $\{B_{1-k}^k, \ldots, B_{n-1}^k\}$ is linearly independent on interval $[t_1, t_n]$ and spans space of all splines of degree k having knots t_i

B-splines, continued

Properties 1 and 2 together say that B-spline functions have local support

Property 3 gives normalization

Property 4 says that they are indeed splines

Property 5 says that for given k these functions form basis for set of all splines of degree k

B-splines, continued

If we use B-spline basis, linear system to be solved for spline coefficients will be nonsingular and banded

Use of B-spline basis yields efficient and stable methods for determining and evaluating spline interpolants, and many library routines for spline interpolation are based on this approach

B-splines are also useful in many other contexts, such as numerical solution of differential equations, as we will see later